rectangular parallolopiped; i. e. length, breadth, height, the diagonals of each of the three different rectangular sides, and the diagonal from an upper corner to the opposite lower corner; or, find integral values, if possible, of a, b, c, d, e, f, and g, as shown in the equations,— $a^2+b^2=c^2$, $a^2+d^2=e^2$, $a^2+f^2=g^2$, $b^2+d^2=f^2$, $b^2+e^2=g^2$, $c^2+d^2=g^2$, $c^2+e^2=f^2$. If not possible, how many of them can have integral values? and which?

Solution by G. B. M ZERR. A. M., Ph. D., Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Let the length, breadth, and height be, $a=8mn(m^4-n^4)$, $b=2mn\{10m^2n^2-3(m^4+n^4)\}$, $c=(m^2-n^2)(m^4+n^4-14m^2n^2)$.

The method of obtaining the above values has been published in several journals and need not be repeated here. From the above we easily get $a^2+b^2=\{2mn(5m^4-6m^2n^2+5n^4)\}^2$, $a^2+c^2=(m^6+17m^4n^2-17m^2n^4-n^6)^2$, $b^2+c^2=(m^6+3m^4n^2+3m^2n^4+n^6)^2=(m^2+n^2)^6$, and $a^2+b^2+c^2=64m^2n^2(m^4-n^4)^2+(m^2+n^2)^6$. This last is a square when $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$. Let $m=pn$. Then must $64p^2(p^2-1)^2+(p^2+1)^4=p^8+68p^6-122p^4+68p^2+1=\square$.

I have not yet succeeded in making this last a square. The edges and diagonals of sides, are integral, satisfying six of the relations.

Let $m=2$, $n=1$; then $a=240$, $b=44$, $c=117$, $\sqrt{a^2+b^2}=244$, $\sqrt{a^2+c^2}=267$, $\sqrt{b^2+c^2}=125$, $\sqrt{a^2+b^2+c^2}=5\sqrt{2929}$.

26. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathemetics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find (1) a *square fraction* the arithmetical difference of whose terms is a *cube*; and (2) find a *cubic fraction* the arithmetical sum of whose terms is a *square*.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1) Let $\frac{y^2}{x^2}$ equal the *square fraction*. Then $x^2-y^2=a$ *cube*$=a^3$. Then $(x+y)(x-y)=a^3$. Put $x+y=a^2$, and $x-y=a$.

$$\text{Then } x=\frac{a^2+a}{2}=\frac{a(a+1)}{2};\text{ and } y=\frac{a^2-a}{2}=\frac{a(a-1)}{2}.$$

Whatever integral values be assigned to a, x and y will always be integral; for whether a is even or odd, $a(a+1)$ and $a(a-1)$ are *even*.

$$\text{Since } x-y=a,\ \frac{a(a-1)}{2}+a=\frac{a(a+1)}{2}.$$

$\therefore$ the denominator of the fraction $\frac{y}{x}$ is always a more than the numerator. Also, as $\frac{a(a-1)}{2}$ is the sum of the series $(1+2+3+\ldots\ldots+a-1)$, $\frac{y}{x}=(1+2+3+\ldots\ldots+a-1)\div[(1+2+3+\ldots\ldots+a-1)+a]$, or $(1+2+3+\ldots\ldots+a-1)\div(1+2+3+\ldots\ldots+a)$. Putting a equal, consecutively, to the successive

integers beginning with *unity*, we have, respectively, $\frac{y}{x}=\frac{0}{1}, \frac{1}{3}, \frac{3}{6}, \frac{6}{10}, \frac{15}{21},$

$\frac{21}{28}, \frac{28}{36}, \frac{36}{45},$ etc., *ad infinitum.*

(2) Let $\frac{m^3}{n^3}$ be the *cubic fraction.* Then $n^3+m^3=\square=b^2$. Then $(n+m)(n^2-nm+m^2)=b^2$. Put $n+m=n^2-nm+m^2=b$.

$$\text{Then } n=\tfrac{1}{2}\left(b+\sqrt{\frac{b(4-b)}{3}}\right), \text{ and } m=\tfrac{1}{2}\left(b-\sqrt{\frac{b(4-b)}{3}}\right).$$

The only integral values of b that will render $\sqrt{\frac{b(4-b)}{3}}$ rational, are 3 and 4. Whence the respective values of n are 2 and 2, and those of m are 1 and 2. $\therefore \frac{m}{n}=\frac{1}{2}$ or $\frac{2}{2}$.

By putting $n+m=\frac{b}{2}$, and $n^2-nm+m^2=2b$ we obtain $\frac{m}{n}=\frac{0}{4}$ and $\frac{4}{8}$.

Other relations of the factors, both in (1) and (2), will yield other results.

II. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics and Science, Mississippi Normal College, Houston, Mississippi.

1st. Let $(x+1)^2=$ one of the terms, and $(a-x)^2$ the other. Then $(a-x)^2-(x+1)^2=$ a cube $=(x+1)^3$. Finding a in terms of x, we get, $a=x+(x+1)\surd(x+2)$, and substituting in the first equation, $[(x+1)\surd(x-2)]^2-(x+1)^2=(x+1)^3$. Now substitute n^2-2 for x and the last equation becomes $(n^2-1)^2n^2-(n^2-1)^2=(n^2-1)^3$.

$$\therefore \text{Fraction}=\frac{n^2(n^2-1)^2}{(n^2-1)^2}, \text{ or } \frac{(n^2-1)^2}{n^2(n^2-1)^2}; \text{ difference}=(n^2-1)^3, \text{ in}$$

which n may be any integer.

2nd. Since both terms must be cubes, we must have

$$n^3(n^3+1)^3+(n^3+1)^3=\text{a square}=(n^3+1)(n^3+1)^3.$$

$$\therefore \text{Fraction}=\frac{n^3(n^3+1)^3}{(n^3+1)^3}, \text{ or } \frac{(n^3+1)^3}{n^3(n^3+1)^3}; \text{ Sum}=(n^3+1)^4, \text{ in which } n$$

may be any integer

III. Solution by H. C. WILKES, Murrayville, West Virginia, and A. H. BELL, Hillsboro, Illinois.

Since when the first number is unity, the sum of any number of successive cubes is a square, if we let $n=$ a root of any cube, then

$$\left[n\left(\frac{n+1}{2}\right)\right]^2-\left[n\left(\frac{n-1}{2}\right)\right]^2=n^3.$$

Let $n=2$, then $3^2-1^2=2^3$,
" $n=3$, then $6^2-3^2=3^3$,
" $n=4$, then $10^2-6^2=4^3$, or in the series 1, 3, 6, 10, 15, etc., we have the difference of the squares of any two contiguous terms equal a cube.

Second case. Let $\frac{y^3}{x^3}$ be the fraction.. Then $x^3+y^3=n^2$, or $(x+y)(x^2-xy+y^2)=n^2$. If $(x+y)$ be a square, then x^2-xy+y^2 will be a square. This is only possible when $x=y$. $\therefore$ the sum of any two equal cubes, the sum of whose roots is a square, will be a square, as $\frac{8^3}{8^3}$, or $\frac{512}{512}$, will be an improper cubic fraction the sum of whose terms will be a square. If $x+y=x^2-xy+y^2$, then x^3+y^3 will be a square. This is only possible when $x=2$, $y=1$, and the proper fraction $\frac{1}{8}$ will be cubic and the sum of the terms a square number.
$\therefore$ $(2^3+1^3)=3^2$; also $2^6(2^3+1^3)=24^2$, etc.

Also solved by *P. S. BERG, A. L. FOOTE, G. B. M. ZERR*, and the *PROPOSER*.

PROBLEMS.

34. Proposed by R H. YOUNG, West Sunbury, Pennsylvania.

Prove (1) that $\frac{n(n+1)(2n+1)}{6}$ is a whole number for all values of n; and (2) prove that $\frac{(n-1)n(n+1)}{24}$ is a whole number when n is odd.

35. Proposed by G. B. M. ZERR, A. M., Ph. D, Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana Texas.

Decompose into the sum of two squares the number $13^2.61^3$.

36. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six integral values of n in $\frac{n(n+1)}{2}=\square$.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

19. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of the circle which is the locus of the middle points of all chords passing through a point taken at random in the surface of a given circle.